Integrable systems related to elliptic branched coverings

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Abstract
The new integrable systems associated with the space of elliptic branched
coverings are constructed. The relationship of these systems with the elliptic
Schlesinger's system (Takasaki 1998 *Lett. Math. Phys.* **44** 143–56) is
described. For the standard two-fold elliptic coverings the integrable system
is written explicitly. The trigonometric degeneration of our construction is
presented.

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1. Introduction

The most well-studied integrable systems such as Korteweg–de Vries, non-linear Schlesinger
and sin-Gordon [2] appear as compatibility conditions of the auxiliary linear system

$$\Phi_x = U\Phi \qquad \Phi_y = V\Phi \tag{1.1}$$

where U, V and Ψ are matrix functions of (x, y) and a constant (i.e. independent of x and y)
spectral parameter $\gamma \in \mathbb{C}$. Matrices U and V for these systems are meromorphic functions of
γ with (x, y)-independent positions of poles.

In 1978, Belinskii and Zakharov [3] and Maison [4] discovered integrability of the Ernst
equation

$$((x-y)G_x G^{-1})_y + ((x-y)G_y G^{-1})_x = 0 \tag{1.2}$$

where $G \in SU(1, 1)/U(1)$, which does not fit into this framework. Namely, the Ernst
equation is a compatibility condition of the system (1.1) with matrices U and V of the form:

$$U = \frac{G_x G^{-1}}{1-\gamma} \qquad V = \frac{G_y G^{-1}}{1+\gamma} \tag{1.3}$$

where the spectral parameter γ is a function of x, y and a 'hidden' ('constant') spectral
parameter λ:

$$\gamma(\lambda, x, y) = \frac{2}{y-x}\left(\frac{x+y}{2} - \lambda + \sqrt{(\lambda-x)(\lambda-y)}\right). \tag{1.4}$$

Therefore, the Ernst equation can be viewed as a 'deformation' of the principal chiral model (PCM) equations. For this model the matrices U and V have the same form (1.3) but γ is a constant (independent of (x, y)) spectral parameter.

The same equation (1.2) for $G \in SU(2)/U(1)$ plays the role of the Gauss–Weingarten system for the so-called Bianchi surfaces in $\mathbb{R}^3$ (surfaces of negative Gaussian curvature of special form [5, 6]).

The general deformation scheme of linear systems of the type (1.1) was proposed in 1989 by Burtsev *et al* [7]. Assuming that the spectral parameter γ in (1.1) depends on x and y, they derived a system of differential equations on γ which provide a part of the compatibility condition of the linear system (1.1). Solutions of the system for γ were found in the recent work [8]; in this work γ is given by the inverse map to the uniformization map of a rational (genus zero) N-fold branched covering of the Riemann sphere when the branch points of the covering are chosen to be independent variables. In other words, a deformation of the linear system (1.1) was associated with the space of rational functions of degree N with simple critical points. In the case of two-fold rational covering, if the matrix dimension equals 2, this scheme leads to the Ernst equation.

In [8] it was also shown how to generalize this approach to the Hurwitz spaces of genus $g \geqslant 2$ (spaces of meromorphic functions on the Riemann surface of genus g) for matrix systems. However, for the genus greater or equal to 2 it is difficult to present any explicit equations. The linear system associated with a genus g branched covering $\mathcal{L}$ has the following form [8]:

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda_m} = U_m \Psi \tag{1.5}$$

where the matrix $U_m(P, \{\lambda_m\})$, $P \in \mathcal{L}$ has only one simple pole at the ramification point P_m of the covering $\mathcal{L}$ and does not have any other singularities. Such a function exists on a genus zero surface, but for the higher genus it must be non-single valued. This means that for genus greater than one the matrices U_m get some multiplicative and (or) additive transformations under tracing along topologically non-trivial cycles of the surface. These transformations depend on branch points of the covering, which makes the corresponding integrable system transcendently non-linear.

In genus one, however, it is possible to develop in detail a scheme analogous to the genus zero case and this is the purpose of the present paper.

Consider the Hurwitz space $H_{1,N}$, the space of N-fold genus one coverings of the Riemann sphere with simple ramification points (coverings consisting of N copies of $\mathbb{C}P^1$ with $2N$ ramification points). Projections of the ramification points on the base of the covering are called the branch points; we assume them to be distinct and denoted by $\lambda_1, \ldots, \lambda_{2N}$. Consider the Abel map $\nu : \mathcal{L} \to \mathbb{C}$ from the genus one covering $\mathcal{L}$ onto its fundamental domain in the complex γ-plane. We denote by $\gamma_1, \ldots, \gamma_{2N}$ the images of the ramification points under this map. They satisfy the following equations as functions of the branch points:

$$\frac{\partial \gamma_n}{\partial \lambda_m} = -\alpha_m [\rho(\gamma_n - \gamma_m) + \rho(\gamma_m)] \qquad m \neq n$$

$$\frac{\partial \gamma_m}{\partial \lambda_m} = \sum_{n=1, n \neq m}^{2N} \alpha_n [\rho(\gamma_m - \gamma_n) + \rho(\gamma_n)]$$

where ρ denotes the logarithmic derivative of the Jacobi theta function θ_1; α_m are some coefficients subject to the differential equations:

$$\frac{\partial \alpha_n}{\partial \lambda_m} = -2\alpha_n \alpha_m \rho'(\gamma_n - \gamma_m)$$

$$\frac{\partial \alpha_m}{\partial \lambda_m} = \sum_{n=1, n \neq m}^{2N} 2\alpha_n \alpha_m \rho'(\gamma_n - \gamma_m).$$

On a covering of genus one, the linear system (1.5) can be written in terms of the elliptic r-matrix, whose transformations under tracing along non-trivial contours of the covering are given by similarity transformations independent of the branch points. Namely, in this paper we consider the linear system (1.5) where matrices U_m look as follows:

$$\overset{1}{U}_m(P) = \overset{2}{\mathrm{tr}}(\overset{12}{r}(\nu(P) - \gamma_m)\overset{2}{J}_m) \tag{1.6}$$

with some matrices $J_m(\{\lambda_k\})$, $P \in \mathcal{L}$. Here we consider all matrices as operators in the tensor product of two copies of $\mathbb{C}^K$: $\overset{1}{A} = A \otimes I, \overset{2}{A} = I \otimes A$; the elliptic r-matrix $\overset{12}{r}$ is a linear operator in $\mathbb{C}^K \otimes \mathbb{C}^K$. The main result of this paper is the integrability of the following system:

$$\frac{\partial \overset{1}{J}_m}{\partial \lambda_n} = -\alpha_n \overset{1}{J}_m \rho'(\gamma_m - \gamma_n) - \alpha_m \overset{2}{\mathrm{tr}}(\overset{12}{r}'(\gamma_m - \gamma_n)\overset{2}{J}_n) - [\overset{1}{J}_m, \overset{2}{\mathrm{tr}}(\overset{12}{r}(\gamma_m - \gamma_n)\overset{2}{J}_n)]. \tag{1.7}$$

It appears as a compatibility condition of the linear system (1.5), (1.6). The systems (1.7) are genus one analogues of the integrable systems constructed in [8]; they give elliptic generalizations of the Ernst equation (1.2).

We define the τ-function for the integrable system (1.7) as follows:

$$\frac{\partial \log \tau}{\partial \lambda_m} = \frac{1}{2\alpha_m} \mathrm{tr}(J_m^2). \tag{1.8}$$

This system is compatible as a corollary of (1.7). For the genus zero two-fold coverings this definition gives rise to one of the metric coefficients on the corresponding spacetime [8].

The non-linear integrable system (1.7), together with the associated linear system (1.5), (1.6), turns out to be closely related to the elliptic Schlesinger system proposed by Takasaki [1]. Namely, from each solution of the elliptic Schlesinger system we can obtain a solution of the system (1.7). For these solutions there is a simple link between τ-function (1.8) and τ-function of the elliptic Schlesinger system:

$$\tau(\{\lambda_m\}) = \prod_{j=1}^{L} \left(\frac{\partial \nu}{\partial \lambda}(Q_j)\right)^{\mathrm{tr}\, A_j^2/2} \tau_{\mathrm{Sch}}(\{z_k\})|_{z_k = \gamma(Q_k)} \tag{1.9}$$

where $\{z_1, \ldots, z_L\}$ is a set of points in the γ-plane which forms a part of monodromy data for the elliptic Schlesinger system; $Q_1, \ldots, Q_L$ are points on the covering whose images under the Abel map ν are given by $z_1, \ldots, z_L$ and whose projection on the λ-sphere do not depend on the branch points $\{\lambda_m\}$; matrices $A_1, \ldots, A_L$ solve the Schlesinger system; the variables $\mathrm{tr}\, A_j^2$ are integrals of the elliptic Schlesinger system.

The paper is organized as follows. In section 2 we discuss the genus zero case and present a slight generalization of the scheme proposed in [8]. In section 3 we derive auxiliary differential equations describing the dependence of the Abel map ν of the genus one covering on the branch points. Further, we introduce the linear system (1.5), (1.6) and derive the integrable system (1.7) as its compatibility condition. Then we define the tau-function of the integrable system. Finally, we write explicitly the system (1.7) in the case of the simplest elliptic covering. Section 4 is devoted to a description of the link of the integrable systems constructed in section 3 with the elliptic Schlesinger system proposed by Takasaki [1]. In section 5 we describe the trigonometric degeneration of the constructed integrable systems (1.7).

2. Integrable systems related to space of rational functions

The goal of this section is to describe integrable systems related to the space of rational functions. We present a different version of the construction proposed in [8]. Consider the space of rational functions of degree N with $2N - 2$ critical points which have the following form:

$$R(\gamma) = \frac{a_N \gamma^N + a_{N-1}\gamma^{N-1} + \cdots + a_0}{\gamma^N + b_{N-1}\gamma^{N-1} + \cdots + b_0}. \tag{2.1}$$

The genus zero algebraic curve

$$\lambda = R(\gamma)$$

can be realized as an N-fold branched covering $\mathcal{L}$ of the λ-sphere $\mathbb{C}P^1$; a point P of the covering is a pair (λ, γ). We denote by π the projection operator from the covering onto the underlying λ-sphere: $\pi(P) = \lambda$. Functions (2.1) have $2N - 2$ critical points counting multiplicities; according to the Riemann–Hurwitz formula, the genus of the corresponding covering $\mathcal{L}$ is zero. We assume the ramification points of the covering to be simple and finite; denote them by $P_1, \ldots, P_{2N-2}$. Their projections $\pi(P_m) = \lambda_m$ on the λ-sphere (the branch points) are critical values of the rational function $R(\gamma)$: $\lambda_m = R(\gamma_m)$, where $\{\gamma_m\}$ are critical points of the function R, i.e. solutions of the equation $R'(\gamma) = 0$. We assume all branch points $\lambda_1, \ldots, \lambda_{2N-2}$ to be distinct.

To each element l of the fundamental group $\pi_1(\mathbb{C}\backslash\{\lambda_1, \ldots, \lambda_{2N-2}\})$ one can assign an element σ_l of the symmetric group S_N, which describes how the sheets of the covering permute when λ goes along the contour l. In this way we can assign to the covering $\mathcal{L}$ a representation of $\pi_1(\mathbb{C}\backslash\{\lambda_1, \ldots, \lambda_{2N-2}\})$ in S_N. For a fixed number of sheets, type of branch points and the assigned representation, the covering is determined by positions of the branch points, i.e. $\{\lambda_m\}_{m=1}^{2N-2}$ gives a set of local coordinates on the Hurwitz space. Observe that this set has $2N-2$ elements whereas the corresponding rational function (2.1) is defined by $2N + 1$ parameters. This is because any Möbius transformation in the γ-sphere (determined by three parameters),

$$\gamma \mapsto \frac{a\gamma + b}{c\gamma + d} \qquad ad - bc = 1 \tag{2.2}$$

leaves the positions of the branch points $\{\lambda_m\}$ invariant.

For our purposes we fix the coefficient a_N in the nominator of the rational function to be a constant, say $a_N = 1$; then the rational function (2.1) becomes

$$R(\gamma) = \frac{\gamma^N + a_{N-1}\gamma^{N-1} + \cdots + a_0}{\gamma^N + b_{N-1}\gamma^{N-1} + \cdots + b_0} \tag{2.3}$$

and at infinity the following asymptotics takes place:

$$\lambda = 1 + \frac{\beta}{\gamma} + o\left(\frac{1}{\gamma}\right) \qquad \text{as} \quad \gamma \sim \infty \tag{2.4}$$

where we denoted $\beta = a_{N-1} - b_{N-1}$.

We shall consider the critical points $\{\gamma_n\}$ of the rational function (2.3) as functions of its critical values $\{\lambda_n\}$. First, note that on the covering $\mathcal{L}$ there is defined a one-to-one function $\nu : \mathcal{L} \to \mathbb{C}P^1$ such that $R(\nu(P)) = \pi(P)$; in particular, the images of ramification points are the critical points of the rational function: $\nu(P_m) = \gamma_m$. The function $\nu(P)$ takes every value only once, thus $\nu(P)$ is holomorphic everywhere except the point which is mapped to infinity; then we can write the expansion of $\nu(P)$ with respect to the local parameter $\sqrt{\lambda - \lambda_m}$ in a neighbourhood of the ramification point P_m (for any $m = 1, \ldots, 2N - 2$) as follows:

$$\nu(P) = \gamma_m + v_m\sqrt{\lambda - \lambda_m} + O(\lambda - \lambda_m) \qquad P \to P_m. \tag{2.5}$$

Let us differentiate these expansions with respect to λ_n and rewrite the result in terms of ν, using the relation $\sqrt{\lambda - \lambda_n} = (\nu - \gamma_n)/\nu_n + O((\nu - \gamma_n)^2)$ which follows from (2.5). We see that the function $\partial \nu / \partial \lambda_n$ is a meromorphic function of ν which has a first-order pole at the point γ_n and is regular at all other critical points, i.e.

$$\frac{\partial \nu}{\partial \lambda_n} = \frac{\alpha_n}{\gamma_n - \nu} + f(\nu) \tag{2.6}$$

where $\alpha_n = (\nu_n/2)^2$, and $f(\nu)$ is a function regular everywhere except the point at infinity. We find the behaviour of this function at infinity differentiating the asymptotics (2.4) (which holds for $\gamma = \nu(P)$ since locally, in a neighbourhood of the pre-image of infinity $P \sim \nu^{-1}(\infty)$, the function $\gamma(\lambda) = \nu(P)$ gives the inverse to $R(\gamma)$, $\gamma \sim \infty$) with respect to λ_n:

$$0 = \frac{\beta_{\lambda_n}}{\nu} - \nu_{\lambda_n} \frac{\beta}{\nu^2} + o\left(\frac{1}{\nu}\right) \qquad \text{as} \quad \nu \sim \infty. \tag{2.7}$$

This implies that the following equations describe the dependence of the function ν on the critical values of the corresponding rational function (2.3):

$$\frac{\partial \nu}{\partial \lambda_n} = \frac{\alpha_n}{\gamma_n - \nu} + \frac{\beta_{\lambda_n}}{\beta}\nu + c_n \qquad n = 1, \ldots, 2N - 2 \tag{2.8}$$

with some functions $c_n = c_n(\{\lambda_k\})$.

The compatibility condition of the system (2.8) gives the following system of differential equations for the critical points $\{\gamma_m\}$ of the rational function (2.3):

$$\frac{\partial \gamma_m}{\partial \lambda_n} = \frac{\alpha_n}{\gamma_n - \gamma_m} + \frac{\beta_{\lambda_n}}{\beta}\gamma_m + c_n \qquad n \neq m. \tag{2.9}$$

Remark. We get the same equations if instead of the rational function (2.3) we consider the one of the form:

$$R(\gamma) = \beta\gamma + \delta + \sum_{k=1}^{N-1} \frac{a_k}{\gamma - b_k} \tag{2.10}$$

which can be obtained from (2.3) by a Möbius transformation.

Consider now the following system of linear differential equations for a matrix-valued function $\Psi(P, \{\lambda_m\})$ $(m = 1, \ldots, 2N - 2)$:

$$\frac{d\Psi}{d\lambda_m}(P) = \frac{\gamma_0 - \gamma_m}{\nu(P) - \gamma_m} G_{\lambda_m} G^{-1} \Psi(P) \tag{2.11}$$

where $\gamma_0 = \nu(P_0)$, the projection $\pi(P_0) = \lambda_0 \in \mathbb{C}P^1$ of the point P_0 is independent of all $\{\lambda_m\}$; $G(\{\lambda_m\})$ is a matrix-valued function. The compatibility condition for (2.11) is given by the following system of non-autonomous (since all γ_m and γ_0 are non-trivial algebraic functions of $\{\lambda_m\}$) coupled PDEs:

$$\left(\frac{\gamma_0 - \gamma_m}{\beta} G_{\lambda_m} G^{-1}\right)_{\lambda_n} = \left(\frac{\gamma_0 - \gamma_n}{\beta} G_{\lambda_n} G^{-1}\right)_{\lambda_m}. \tag{2.12}$$

The described construction of the integrable systems gives a realization of the scheme of Burtsev *et al* [7] who derived the compatibility conditions for the deformed linear system of the type (1.1). They obtained differential equations on the variable spectral parameter of the linear system which form a part of the compatibility condition. It was shown in [8] that the function $\nu(P)$ is a solution of these differential equations.

In the case of the two-fold coverings ($N = 2$) corresponding to the rational function of the form (2.10) with $\beta = 1, \delta = 0$ (the normalization considered in [8]) system (2.12) coincides with the Ernst equation (1.2) after the identification $\lambda_1 = x, \lambda_2 = y$ (see [8]).

There exists a well-known relationship between these rational two-fold coverings and the surface theory: the Gauss–Weingarten equation for a surface in $\mathbb{R}^3$ with the Gaussian curvature $K = -[\rho(x, y)]^{-2}$ can be written in the following form [6]:

$$(\rho G_x G^{-1})_y + (\rho G_y G^{-1})_x = 0 \tag{2.13}$$

for $G \in SU(2)/U(1)$, which for the case of the Bianchi surfaces ($\rho(x, y) = x - y$) formally coincides with equation (1.2).

Here the natural question arises: are there other coverings for which the system (2.12) takes the form of the Gauss–Weingarten equation for some surfaces? (Then it would be a new integrable case in surface theory.) This occurs if the system (2.12) has the property $\gamma_0 - \gamma_m = -(\gamma_0 - \gamma_n)$ for some pair of indices m, n; that is

$$\frac{\gamma_m + \gamma_n}{2} = \gamma_0 \tag{2.14}$$

where $\gamma_0 = \nu(P_0)$ is the image of the point $P_0 \in \mathcal{L}$ whose projection λ_0 on the λ-sphere does not depend on $\{\lambda_k\}$. Existence of such systems is an open question. Since the covering is locally defined by $2N - 2$ independent variables $\{\lambda_m\}$, two additional parameters of the rational function (2.3) could be used to impose some relations on $\{\gamma_m\}$. As was already noted, the freedom to choose these parameters corresponds to two Möbius transformations in the γ-sphere: $\gamma \to a\gamma$ and $\gamma \to \gamma + b$. But the condition (2.14) is invariant with respect to both of these transformations, which means that for the given degree N of a rational function we do not have any freedom to impose condition (2.14) for any pair of m and n. However, there is still a possibility that (2.14) holds for some rational coverings as in the case of $N = 2$.

3. Integrable systems related to elliptic branched coverings

In this section we construct an elliptic analogue of the integrable system (2.12).

3.1. Differential equations for images of ramification points of elliptic coverings in the fundamental domain

The Hurwitz space $H_{1,N}$ is the space of meromorphic functions of degree N on Riemann surfaces of genus one. Consider a meromorphic double-periodic function R of $\gamma \in \mathbb{C}$ with periods 1 and μ and N simple poles within the fundamental domain $T = \mathbb{C}/\{1, \mu\}$. As a function on T, $R(\gamma)$ has degree N. The equation

$$\lambda = R(\gamma) \tag{3.1}$$

defines an N-fold branched covering (we again call it $\mathcal{L}$) of the Riemann sphere. A point P of the covering is a pair: $P = (\lambda, \gamma)$. According to the Riemann–Hurwitz formula, this covering has $2N$ ramification points counting multiplicities; we assume them to be simple and finite and denote by $P_1, \ldots, P_{2N}$. Projections $\{\pi(P_m)\}$ of the ramification points onto the λ-sphere (the base of the covering) are called the branch points. They are given by critical values $\lambda_1, \ldots, \lambda_{2N}$ of the meromorphic function $R(\gamma) : \lambda_m = R(\hat{\gamma}_m)$, where $\hat{\gamma}_1, \ldots, \hat{\gamma}_{2N}$ are critical points of $R(\gamma)$, solutions of the equation $R'(\gamma) = 0$. We assume the branch points to be distinct: $\lambda_m \neq \lambda_n$ for $m \neq n$. Our choice of the local parameters on $\mathcal{L}$ is standard: in a neighbourhood of a ramification point P_m we take $x(P) = \sqrt{\lambda - \lambda_m}, P \in \mathcal{L}, P \sim P_m$;

in a neighbourhood of a point at infinity on any sheet we take $x = 1/\lambda$; at any other point variable λ itself is used as a local coordinate. To the covering $\mathcal{L}$ it is assigned a representation of the fundamental group $\pi_1(\mathbb{C}\backslash\{\lambda_1, \ldots, \lambda_{2N}\})$ in the following way. To each element l of the fundamental group one can assign an element σ_l of the symmetric group S_N, which describes how the sheets permute when λ goes along the contour l on the base of the covering. We fix this representation of $\pi_1(\mathbb{C}\backslash\{\lambda_1, \ldots, \lambda_{2N}\})$. Then for a fixed number of sheets and type of ramification points (we fix them to be simple) the covering is determined by positions of the branch points; thus we consider $\{\lambda_m\}_{m=1}^{2N}$ as a set of local coordinates on the space of elliptic coverings.

Introduce on $\mathcal{L}$ some canonical basis of cycles (a, b). Denote by $\mathbf{v}(P)$, $P \in \mathcal{L}$, the holomorphic Abelian differential with normalized a-period:

$$\oint_a \mathbf{v} = 1. \tag{3.2}$$

For our covering it has the form

$$\mathbf{v}(P) = \frac{d\lambda}{R'(\gamma)} = d\gamma. \tag{3.3}$$

The integral over the b-cycle gives the module μ of the elliptic Riemann surface $\mathcal{L}$:

$$\mu = \oint_b \mathbf{v}. \tag{3.4}$$

The function $\nu(P)$ which maps $\mathcal{L}$ onto the fundamental domain $T = \mathbb{C}/\{1, \mu\}$ is given by the Abel map

$$\nu(P) = \int_{\infty^{(0)}}^{P} \mathbf{v} \tag{3.5}$$

where we choose the initial point of integration to coincide with the point at infinity on some (the zeroth) sheet of the covering $\mathcal{L}$. We denote the images of the ramification points under this map by γ_m. They differ from the critical points $\{\hat{\gamma}_m\}$ of the function R by a shift (corresponding to the choice of initial point of integration in (3.5)) modulo the period lattice $\{k\mu + l; l, k \in \mathbb{N}\}$:

$$\gamma_m \equiv \hat{\gamma}_m - c \qquad m = 1, \ldots, 2N \tag{3.6}$$

where c is the second coordinate of the point $\infty^{(0)} \in \mathcal{L} : \infty^{(0)} = (\infty, c)$.

The Jacobi theta functions are given by

$$\theta[p, q](\gamma; \mu) = \sum_{m \in \mathbb{Z}} \exp\{\pi i \mu(m + p)^2 + 2\pi i(m + p)(\gamma + q)\}. \tag{3.7}$$

We denote by $\rho(\gamma)$ the logarithmic derivative of theta-function $\theta_1(\gamma) = -\theta[1/2, 1/2](\gamma)$:

$$\rho(\gamma) = \frac{d}{d\gamma} \log \theta_1(\gamma). \tag{3.8}$$

It has the following periodicity properties:

$$\rho(\gamma + 1) = \rho(\gamma) \qquad \rho(\gamma + \mu) = \rho(\gamma) - 2\pi i. \tag{3.9}$$

The derivative $\rho'(\gamma)$ coincides with the Weierstrass $\mathcal{P}$-function up to a rescaling of the argument and an additive constant.

The following theorem describes the dependence of the map $\nu(P)$ (3.5) on λ and the branch points $\{\lambda_m\}$; it provides an elliptic version of equations (2.8).

Theorem 1. *The function $v(\lambda, \{\lambda_m\})$ defined by (3.5) satisfies the following system of differential equations:*

$$\frac{\partial v}{\partial \lambda} = \sum_{k=1}^{2N} \alpha_k [\rho(v - \gamma_k) + \rho(\gamma_k)] \tag{3.10}$$

$$\frac{\partial v}{\partial \lambda_m} = -\alpha_m [\rho(v - \gamma_m) + \rho(\gamma_m)] \qquad m = 1, \ldots, 2N \tag{3.11}$$

where we denoted

$$\alpha_m = \frac{1}{2} v_m^2 = \frac{1}{2} \left[\frac{\mathbf{v}(P)}{d\sqrt{\lambda - \lambda_m}} \Bigg|_{P=P_m} \right]^2. \tag{3.12}$$

$\{\gamma_m\}$ *are the images of the ramification points under the map* $v : \gamma_m = v(P_m)$.

Remark. The form (3.3) of the holomorphic normalized differential implies that $\alpha_m = [R''(\hat{\gamma}_m)]^{-1}$.

Proof of theorem 1. From (3.5) we see that the function $v(P)$ is holomorphic in a neighbourhood of the ramification point P_m and behaves as follows:

$$v(P) = \gamma_m + v_m\sqrt{\lambda - \lambda_m} + \mathcal{O}(\lambda - \lambda_m) \qquad \text{as} \quad P \to P_m \tag{3.13}$$

where $\sqrt{\lambda - \lambda_m}$ is the local coordinate in a neighbourhood of P_m, and v_m is defined by (3.12). Therefore, in this neighbourhood

$$\frac{\partial v}{\partial \lambda}(P) = \frac{v_m}{2\sqrt{\lambda - \lambda_m}} + \mathcal{O}(1) \tag{3.14}$$

$$\frac{\partial v}{\partial \lambda_m}(P) = -\frac{v_m}{2\sqrt{\lambda - \lambda_m}} + \mathcal{O}(1) \tag{3.15}$$

$$\frac{\partial v}{\partial \lambda_n}(P) = \mathcal{O}(1) \qquad n \neq m. \tag{3.16}$$

We rewrite these expansions in terms of the coordinate v taking into account definition (3.12) of α_m, and the correspondence between local parameters $v - \gamma_m$ and $\sqrt{\lambda - \lambda_m}$ is given by (3.13)

$$\frac{\partial v}{\partial \lambda}(P) = \frac{\alpha_m}{v - \gamma_m} + \mathcal{O}(1) \qquad \frac{\partial v}{\partial \lambda_n}(P) = -\delta_{mn}\frac{\alpha_m}{v - \gamma_m} + \mathcal{O}(1) \tag{3.17}$$

as $P \to P_m$.

The function $v(P)$ transforms as follows under the tracing along basic cycles on $\mathcal{L}$:

$$v(P^a) = v(P) + 1 \qquad v(P^b) = v(P) + \mu \tag{3.18}$$

where $v(P^a), v(P^b)$ denote the analytic continuation of $v(P)$ along a- and b-cycles, respectively. Therefore, the derivative v_λ is periodic with respect to tracing along the basic cycles. Then the function v_λ has periods 1 and μ in the γ-plane. Its local behaviour at the points $\gamma_m, m = 1, \ldots, 2N$ is given by (3.17). Hence, we conclude that $\sum_{k=1}^{2N} \alpha_k = 0$ as sum of residues, and the derivative v_λ can be expressed as follows in terms of function ρ:

$$\frac{\partial v}{\partial \lambda}(v) = \sum_{k=1}^{2N} \alpha_k \rho(v - \gamma_k) + \text{const.} \tag{3.19}$$

To determine the constant in (3.19) consider a neighbourhood of $P = \infty^{(0)}$. The Abel map (3.5) is zero at this point, $\nu(\infty^{(0)}) = 0$, and we can write its behaviour there as follows:

$$\nu(\lambda) = \frac{\alpha}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \qquad \text{as} \quad \lambda \to \infty. \tag{3.20}$$

(Note that $\alpha \neq 0$ since we assume $\infty^{(0)}$ not to be a ramification point.) Therefore, for the λ-derivative we have in a neighbourhood of $P = \infty^{(0)}$:

$$\frac{\partial \nu}{\partial \lambda}(\lambda) = -\frac{\alpha}{\lambda^2} + \mathcal{O}\left(\frac{1}{\lambda^3}\right) \qquad \lambda \to \infty.$$

Rewriting as before this expansion in terms of the coordinate ν ((3.20) implies $\lambda \sim \alpha/\nu$) we see that $\nu_\lambda(\nu = 0) = 0$:

$$\frac{\partial \nu}{\partial \lambda}(\nu) = -\frac{\nu^2}{\alpha} + \mathcal{O}(\nu^3) \qquad \text{as} \quad \nu(P) \to 0.$$

Therefore, (3.19) turns into (3.10).

Consider now ν_{λ_m}. In the γ-plane it has only one simple pole at $\nu = \gamma_m$ as follows from (3.17). The periodicity properties (3.18) of the Abel map imply that

$$\frac{\partial \nu}{\partial \lambda_m}(\nu + 1) = \frac{\partial \nu}{\partial \lambda_m}(\nu) \qquad \text{and} \qquad \frac{\partial \nu}{\partial \lambda_m}(\nu + \mu) = \frac{\partial \nu}{\partial \lambda_m}(\nu) + \frac{\partial \mu}{\partial \lambda_m}. \tag{3.21}$$

The function $-\alpha_m \rho(\nu - \gamma_m) + \text{const}$ satisfies the periodicity condition (3.21) since, due to the Rauch variational formulae [9], we have

$$\frac{\partial \mu}{\partial \lambda_m} = \pi i \nu_m^2 = 2\pi i \alpha_m. \tag{3.22}$$

To find the constant term we again put $\nu = 0$, i.e. $P = \infty^{(0)}$. Then from the asymptotics (3.20) we see that $\nu_{\lambda_m}(\nu = 0) = 0$, which leads to (3.11). $\square$

Remark. Equations (3.11) can also be deduced from the Rauch variational formulae for the differential **v** [9].

Compatibility conditions of the system (3.10), (3.11) imply the system of differential equations describing the dependence of $\{\gamma_m\}$ on the branch points $\{\lambda_m\}$ (the indices run through the set $\{1, \ldots, 2N\}$):

$$\frac{\partial \gamma_n}{\partial \lambda_m} = -\alpha_m[\rho(\gamma_n - \gamma_m) + \rho(\gamma_m)] \qquad m \neq n \tag{3.23}$$

$$\frac{\partial \gamma_m}{\partial \lambda_m} = \sum_{k=1, k \neq m}^{2N} \alpha_k[\rho(\gamma_m - \gamma_k) + \rho(\gamma_k)]. \tag{3.24}$$

The equations for residues α_m which also follow from the compatibility of (3.10) and (3.11) look as follows:

$$\frac{\partial \alpha_n}{\partial \lambda_m} = -2\alpha_n \alpha_m \rho'(\gamma_n - \gamma_m) \tag{3.25}$$

$$\frac{\partial \alpha_m}{\partial \lambda_m} = \sum_{k=1, k \neq m}^{2N} 2\alpha_k \alpha_m \rho'(\gamma_k - \gamma_m). \tag{3.26}$$

In fact, equations (3.25) and (3.26) are nothing but the Rauch variational formulae [9] for the holomorphic differential **v**.

3.2. Integrable systems

Denote the matrix dimension of our system by K. The classical elliptic r-matrix is the following linear operator in the tensor product of two copies of $\mathbb{C}^K$:

$$\overset{12}{r}(\gamma) = \sum_{\substack{A,B=0 \\ (A,B)\neq(0,0)}}^{K-1} w_{AB}(\gamma)\overset{1}{\sigma}_{AB}\overset{2}{\sigma}{}^{AB} \tag{3.27}$$

where w_{AB} are given by the combinations of Jacobi's theta functions (3.7) $((A, B) \neq (0, 0))$:

$$w_{AB}(\gamma) = \frac{\theta_{[AB]}(\gamma)\theta'_{[00]}(0)}{\theta_{[AB]}(0)\theta_{[00]}(\gamma)} \tag{3.28}$$

where we denote

$$\theta_{[AB]}(\gamma) = \theta_{[AB]}(\gamma; \mu) = \theta\left[\frac{A}{K} - \frac{1}{2}, \frac{1}{2} - \frac{B}{K}\right](\gamma; \mu).$$

All w_{AB} have a simple pole with unit residue at $\gamma = 0$ and the following twist properties:

$$w_{AB}(\gamma + 1) = \epsilon^A w_{AB}(\gamma) \qquad w_{AB}(\gamma + \mu) = \epsilon^B w_{AB}(\gamma) \tag{3.29}$$

where $\epsilon = e^{2\pi i/K}$. The matrices σ_{AB} are the higher rank analogues of the Pauli matrices; they form a basis of $sl(K, \mathbb{C})$ and are defined as follows (for $(A, B) \neq (0, 0)$):

$$\sigma_{AB} = H^A F^B \tag{3.30}$$

where F is the diagonal matrix

$$F = \mathrm{diag}\{1, \epsilon, \epsilon^2, \dots, \epsilon^{K-1}\}$$

and H is the permutation matrix

$$H = \begin{pmatrix} 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1 \\ 1 & 0 & . & . & . & 0 \end{pmatrix}.$$

These matrices satisfy the relations $\epsilon F H = H F$, and $F^K = H^K = I$.

Together with σ_{AB} we introduce the dual basis σ^{AB}:

$$\sigma^{AB} = \frac{\epsilon^{-AB}}{K}\sigma_{-A,-B} \tag{3.31}$$

such that

$$\mathrm{tr}(\sigma_{AB}\sigma^{CD}) = \delta_A^C\delta_B^D. \tag{3.32}$$

From (3.29) and properties of matrices F and H we derive the following periodicity properties of the elliptic r-matrix (3.27):

$$\overset{12}{r}(\gamma + 1) = \overset{1}{F}{}^{-1}\overset{12}{r}(\gamma)\overset{1}{F} \qquad \overset{12}{r}(\gamma + \mu) = \overset{1}{H}\overset{12}{r}(\gamma)\overset{1}{H}{}^{-1}. \tag{3.33}$$

In the following we shall also need the functions

$$\mathcal{Z}_{AB}(\gamma) = \frac{w_{AB}(\gamma)}{2\pi i}\left(\frac{\theta'_{[AB]}(\gamma)}{\theta_{[AB]}(\gamma)} - \frac{\theta'_{[AB]}(0)}{\theta_{[AB]}(0)}\right) \qquad (A, B) \neq (0, 0) \tag{3.34}$$

which have no singularities and transform as follows:

$$\mathcal{Z}_{AB}(\gamma + 1) = \epsilon^A \mathcal{Z}_{AB} \qquad \mathcal{Z}_{AB}(\gamma + \mu) = \epsilon^B (\mathcal{Z}_{AB}(\gamma) - w_{AB}(\gamma)). \quad (3.35)$$

Using the fact that theta functions satisfy the heat equation,

$$\frac{\partial^2 \theta[p,q](\gamma;\mu)}{\partial \gamma^2} = 4\pi i \frac{\partial \theta[p,q](\gamma,\mu)}{\partial \mu} \qquad (3.36)$$

we get the the following relation between $\mathcal{Z}_{AB}$ and w_{AB}:

$$\partial_\mu w_{AB}(\gamma;\mu) = \partial_\gamma \mathcal{Z}_{AB}(\gamma;\mu). \qquad (3.37)$$

Now we are in a position to write down an 'elliptic' counterpart of the linear system (2.11):

$$\frac{d\overset{1}{\Psi}(P)}{d\lambda_m} = \overset{2}{\mathrm{tr}}(\overset{12}{r}(\nu(P) - \gamma_m)\overset{2}{J_m})\overset{1}{\Psi}(P) \qquad (3.38)$$

where

$$J_m = \sum_{\substack{A,B=0 \\ (A,B) \neq (0,0)}}^{K-1} J_m^{AB}\sigma_{AB}$$

with scalars J_m^{AB}. Here $m = 1,\ldots,2N$, $\Psi = \Psi(P,\{\lambda_m\})$ is a matrix-valued function; as before, $\nu(P)$ is the Abel map (3.5) from the covering $\mathcal{L}$ onto its fundamental domain $T = \mathbb{C}/\{1,\mu\}$; $\gamma_m = \nu(P_m)$. The compatibility condition of this system

$$(\overset{2}{\mathrm{tr}}(\overset{12}{r}(\nu(P) - \gamma_m)\overset{2}{J_m}))_{\lambda_n} - (\overset{2}{\mathrm{tr}}(\overset{12}{r}(\nu(P) - \gamma_n)\overset{2}{J_n}))_{\lambda_m}$$

$$+ [\overset{2}{\mathrm{tr}}(\overset{12}{r}(\nu(P) - \gamma_m)\overset{2}{J_m}), \overset{2}{\mathrm{tr}}(\overset{12}{r}(\nu(P) - \gamma_n)\overset{2}{J_n})] = 0 \qquad (3.39)$$

gives the system of differential equations for matrices J_m as functions of the branch points λ_m:

$$\frac{\partial \overset{1}{J_m}}{\partial \lambda_n} = -\alpha_n \overset{1}{J_m}\rho'(\gamma_m - \gamma_n) - \alpha_m \overset{2}{\mathrm{tr}}(\overset{12}{r}{}'(\gamma_m - \gamma_n)\overset{2}{J_n})$$

$$- [\overset{1}{J_m}, \overset{2}{\mathrm{tr}}(\overset{12}{r}(\gamma_m - \gamma_n)\overset{2}{J_n})] \qquad m \neq n \qquad (3.40)$$

where r' stands for the derivative of the r-matrix with respect to its argument. To prove that the compatibility condition reduces to (3.40) we, first, compute the derivatives in (3.39) using the chain rule:

$$r_{\lambda_n}(\gamma) = r_\mu(\gamma)\mu_{\lambda_n} + r'(\gamma)\gamma_{\lambda_n}.$$

The derivative of the period μ is given by (3.22); for differentiation of ν and $\{\gamma_m\}$ one uses the equations (3.11) and (3.23), respectively. Then we note that the vector bundle χ over the Riemann surface $\mathcal{L}$, whose monodromy matrices along the cycles a and b are given by F^{-1} and H, respectively, is stable [10]. Checking the periodicity properties of the left-hand side of (3.39) we see that it is a section of the adjoint bundle ad χ. Due to the stability of χ the bundle ad χ does not have holomorphic sections (see, for example, [11]). Therefore, for condition (3.39) to hold it suffices that the left-hand side has no singularities; this is equivalent to the system (3.40).

Equations (3.40) form the non-autonomous non-linear integrable system associated with the space of elliptic coverings which gives an elliptic analogue of the integrable system (2.12).

3.3. Tau-function

Let us introduce an object which we shall call the tau-function of the system (3.40):

$$\frac{\partial \log \tau}{\partial \lambda_m} = \frac{1}{2\alpha_m} \, \mathrm{tr}\big(J_m^2\big). \tag{3.41}$$

To prove consistency of the definition we compute the derivatives of the right-hand side, $\frac{\partial}{\partial \lambda_n}\big(\frac{1}{2\alpha_m}\,\mathrm{tr}\big(J_m^2\big)\big)$, using (3.40). Then we get

$$\frac{\partial^2 \log \tau}{\partial \lambda_m \partial \lambda_n} = -\overset{1}{\mathrm{tr}}\,\overset{2}{\mathrm{tr}}\,(\overset{1}{J_m}\,\overset{2}{J_n}\,\overset{12}{r}{}'(\gamma_m - \gamma_n)).$$

This expression is symmetric in m and n, due to the following properties of the r-matrix:

$$\overset{12}{r}(\gamma) = -\overset{21}{r}(-\gamma)$$

and

$$\overset{12}{r}{}'(\gamma) = \overset{21}{r}{}'(-\gamma).$$

This proves compatibility of the equations (3.41).

An alternative definition of the tau-function (3.41) can be given in terms of the one form $\mathrm{d}\Psi\,\Psi^{-1} = \Psi_\nu\Psi^{-1}\,\mathrm{d}\nu$:

$$\frac{\partial \log \tau}{\partial \lambda_m} = \frac{1}{2}\,\mathrm{res}|_{P_m}\left\{\frac{\mathrm{tr}(\mathrm{d}\Psi\,\Psi^{-1})^2}{\mathrm{d}\lambda}\right\}. \tag{3.42}$$

To prove the equivalence of the two definitions, first note that we can write

$$\mathrm{d}\lambda = \frac{\partial \lambda}{\partial \nu}\,\mathrm{d}\nu. \tag{3.43}$$

Therefore using (3.10) for $\partial\nu/\partial\lambda$ we get

$$\frac{\mathrm{tr}(\mathrm{d}\Psi\,\Psi^{-1})^2}{\mathrm{d}\lambda} = \frac{\partial\nu}{\partial\lambda}\,\mathrm{tr}(\Psi_\nu\Psi^{-1})^2\,\mathrm{d}\nu = \left(\sum_{k=1}^{2N}\alpha_k(\rho(\nu-\gamma_k)+\rho(\gamma_k))\right)\mathrm{tr}(\Psi_\nu\Psi^{-1})^2\,\mathrm{d}\nu. \tag{3.44}$$

Further, we write the 'full' derivative of Ψ with respect to λ_m as follows:

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda_m} = \frac{\partial\Psi}{\partial\lambda_m} + \frac{\partial\nu}{\partial\lambda_m}\frac{\partial\Psi}{\partial\nu} \tag{3.45}$$

then using the form of the linear system (3.38) and formula (3.11) for the derivative of ν, we rewrite (3.45) in the form:

$$\overset{2}{\mathrm{tr}}(\overset{12}{r}(\nu-\gamma_m)\,\overset{2}{J_m}) = \frac{\partial\overset{1}{\Psi}}{\partial\lambda_m}\,\overset{1}{\Psi}{}^{-1} - \alpha_m(\rho(\nu-\gamma_m)+\rho(\gamma_m))\frac{\partial\overset{1}{\Psi}}{\partial\nu}\,\overset{1}{\Psi}{}^{-1} \tag{3.46}$$

from which one can find $\mathrm{tr}(\Psi_\nu\Psi^{-1})^2$ and see that (3.42) is equivalent to (3.41).

3.4. Integrable system in the case of two-fold elliptic coverings

The simplest elliptic covering $\mathcal{L}$ has two sheets and four ramification points. It corresponds to the hyperelliptic curve given by the following equation:

$$\omega^2 = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$$

where λ_m, $m = 1, \ldots, 4$, are branch points. On the covering we choose the basic cycle a to encircle ramification points P_1, P_2, and the b-cycle to encircle points P_2 and P_3. For this Riemann surface the normalized holomorphic differential $\mathbf{v}$ is given by

$$\mathbf{v} = \frac{\mathrm{d}\lambda}{\omega}\left[\oint_a \frac{\mathrm{d}\lambda}{\omega}\right]^{-1}. \tag{3.47}$$

As before, μ is the b-period of the surface $\mathcal{L} : \mu = \oint_b \mathbf{v}(P)$. Consider the map $\tilde{v}$ from the covering $\mathcal{L}$ onto its fundamental domain $T = \mathbb{C}/\{1, \mu\}$:

$$\tilde{v}(P) = \int_{P_1}^{P} \mathbf{v}(P).$$

This map differs from the map $v(P)$ (3.5) by a function of branch points:

$$v(P) = \tilde{v}(P) + h(\{\lambda_m\})$$

where $h(\{\lambda_m\}) = \int_{\infty^{(0)}}^{P_1} \mathbf{v}(P)$. For our choice of basic cycles the images $\tilde{\gamma}_m$ of ramification points under the map $\tilde{v}$ are given by

$$\tilde{\gamma}_1 = \tilde{v}(P_1) = 0 \qquad\qquad \tilde{\gamma}_2 = \tilde{v}(P_2) = \frac{1}{2}$$

$$\tilde{\gamma}_3 = \tilde{v}(P_3) = \frac{1}{2} + \frac{\mu}{2} \qquad \tilde{\gamma}_4 = \tilde{v}(P_4) = \frac{\mu}{2}.$$

Since $\gamma_m - \gamma_n = \tilde{\gamma}_m - \tilde{\gamma}_n$ (where $\gamma_m = v(P_m)$, $m = 1, \ldots, 4$ are as before the images of ramification points under the map v (3.5)), we can use these values of $\{\tilde{\gamma}_m\}$ to write explicitly the system (3.40) for the simplest covering. To do this we also calculate the coefficients $\{\alpha_m\}_{m=1}^4$ defined by (3.12). The form (3.47) of the normalized holomorphic differential $\mathbf{v}$ implies:

$$\alpha_1 = \frac{v_1^2}{2} = \frac{2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)A^2}$$

where $A = \oint_a \frac{d\lambda}{\omega}$. From the Thomae formulae [12] we see that

$$A^2 = \frac{4\pi^2\theta_4^4}{(\lambda_1 - \lambda_4)(\lambda_3 - \lambda_2)}$$

and therefore we have the following expressions for the coefficients α_m:

$$\alpha_1 = \frac{\lambda_3 - \lambda_2}{2\pi^2\theta_4^4(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \qquad \alpha_2 = -\frac{\lambda_1 - \lambda_4}{2\pi^2\theta_4^4(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_4)}$$

$$\alpha_3 = \frac{\lambda_1 - \lambda_4}{2\pi^2\theta_4^4(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_4)} \qquad \alpha_4 = -\frac{\lambda_3 - \lambda_2}{2\pi^2\theta_4^4(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3)}.$$

Now we can write down the integrable system (3.40) explicitly for $K = 2$ (K is the matrix dimension of the system). In this case we use the standard Pauli basis $\{\sigma_1, \sigma_2, \sigma_3\}$ related to the matrices σ_{AB} as follows:

$$\sigma_{10} = \sigma_1 \qquad\qquad \sigma_{11} = i\sigma_2 \qquad\qquad \sigma_{01} = \sigma_3$$

$$\sigma^{10} = \frac{1}{2}\sigma_1 \qquad \sigma^{11} = \frac{i}{2}\sigma_2 \qquad \sigma^{01} = \frac{1}{2}\sigma_3. \tag{3.48}$$

The corresponding notation for components of J_m is

$$J_m^1 = J_m^{10} \qquad J_m^2 = iJ_m^{11} \qquad J_m^3 = J_m^{01}. \tag{3.49}$$

We shall write the equations for $(J_1)_{\lambda_2}$ $\left(J_1 = J_1^1\sigma_1 + J_1^2\sigma_2 + J_1^3\sigma_3\right)$. The remaining equations for $(J_m)_{\lambda_n}$ in the case of two-fold elliptic covering have a similar form.

$$\frac{\partial J_1^1}{\partial\lambda_2} = \frac{\lambda_1 - \lambda_4}{2\pi^2(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_4)} J_1^1 \frac{1}{\theta_4^4}\frac{\theta_2''}{\theta_2} + 2\pi i J_1^3 J_2^2\theta_4^2$$

$$\frac{\partial J_1^2}{\partial\lambda_2} = \frac{\lambda_1 - \lambda_4}{2\pi^2(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_4)} J_1^2 \frac{1}{\theta_4^4}\frac{\theta_2''}{\theta_2} - 2\pi i J_1^3 J_2^1\theta_3^2$$

$$\frac{\partial J_1^3}{\partial\lambda_2} = \frac{\lambda_1 - \lambda_4}{2\pi^2(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_4)} J_1^3 \frac{1}{\theta_4^4}\frac{\theta_2''}{\theta_2} + \frac{\lambda_3 - \lambda_2}{2(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} J_2^3\frac{\theta_3^2}{\theta_4^2}$$

$$+ 2\pi i \left(J_1^2 J_2^1\theta_3^2 - J_1^1 J_2^2\theta_4^2\right).$$

Here $\theta_2 = \theta\left[\frac{1}{2}, 0\right](0)$; $\theta_3 = \theta[0, 0](0)$; $\theta_4 = \theta\left[0, \frac{1}{2}\right](0)$ and $\theta_2'' = \theta_2''(0)$ are the standard theta-constants.

4. Relationship to the Schlesinger system

The elliptic Schlesinger system [1] describes isomonodromic deformations of solutions $\Psi(\gamma, \{z_i\})$ of the following matrix linear differential equation:

$$\frac{d\Psi}{d\gamma} = A(\gamma)\Psi \tag{4.1}$$

where γ is a coordinate on the torus $T = \mathbb{C}/\{1, \mu\}$; $A(\gamma)$ is a meromorphic $sl(K, \mathbb{C})$-valued matrix:

$$\overset{1}{A}(\gamma) = \sum_{j=1}^{L} \overset{2}{\mathrm{tr}}(\overset{12}{r}(\gamma - z_j)\overset{2}{A}_j).$$

$r(\gamma)$ is the elliptic r-matrix (3.27); $z_j \in T$, $j = 1, \ldots, L$; L is some integer. At the points $\{z_j\}$ the matrix $A(\gamma)$ has simple poles with residues A_j. The residues are, in turn, parametrized as follows:

$$A_j = \sum_{\substack{A, B=0 \\ (A,B) \neq (0,0)}}^{K-1} A_j^{AB} \sigma_{AB} \tag{4.2}$$

where matrices σ_{AB} are given by (3.30); $A_j^{AB} \in \mathbb{C}$. The matrix $A(\gamma)$ has the following periodicity properties:

$$A(\gamma + 1) = F^{-1}A(\gamma)F \qquad A(\gamma + \mu) = HA(\gamma)H^{-1}.$$

It is assumed that Ψ has asymptotical expansion near z_j, $j = 1, \ldots, L$, of the form:

$$\Psi(\gamma) = (G_j + O(\gamma - z_j))(\gamma - z_j)^{T_j} C_j \tag{4.3}$$

where matrices G_j, C_j, T_j do not depend on γ; $C_j, G_j \in SL(K, \mathbb{C})$, and T_j are diagonal traceless matrices such that any two entries of T_j do not differ by an integer number. The function Ψ transforms as follows with respect to periods 1 and μ of the torus T:

$$\Psi(\gamma + 1) = F^{-1}\Psi(\gamma)M_a \qquad \Psi(\gamma + \mu) = H\Psi(\gamma)M_b$$

and being analytically continued along a contour l_j surrounding the point z_j the function Ψ gains a right multiplier:

$$\Psi(\gamma^{l_j}) = \Psi(\gamma)M_j$$

where M_a, M_b, M_j are called the monodromy matrices. The assumption of independence of all monodromy matrices of the positions of singularities $\{z_j\}$ and the b-period μ of the elliptic Riemann surface is called the isomonodromy condition. This condition together with expansion (4.3) gives the following dependence of Ψ on μ and $\{z_j\}_{j=1}^{L}$:

$$\overset{1}{\Psi}_{z_i}\overset{1}{\Psi}^{-1} = -\overset{2}{\mathrm{tr}}(\overset{12}{r}(\gamma - z_i)\overset{2}{A}_i) \tag{4.4}$$

$$\Psi_\mu \Psi^{-1} = \sum_{j=1}^{L} \sum_{\substack{A, B=0 \\ (A,B) \neq (0,0)}}^{K-1} A_j^{AB} \mathcal{Z}_{AB}(\gamma - z_i)\sigma_{AB} \tag{4.5}$$

the functions $\mathcal{Z}_{AB}$ were defined by (3.34). The compatibility condition of (4.4), (4.5) and (4.1) gives the Schlesinger system on the elliptic surface:

$$\frac{\partial \overset{1}{A_i}}{\partial z_j} = [\overset{1}{A_i}, \overset{2}{\text{tr}}(\overset{12}{r}(z_i - z_j)\overset{2}{A_j})] \qquad i \neq j$$

$$\frac{\partial \overset{1}{A_i}}{\partial z_i} = - \sum_{j=1, j\neq i}^{L} [\overset{1}{A_i}, \overset{2}{\text{tr}}(\overset{12}{r}(z_i - z_j)\overset{2}{A_j})]$$

$$\frac{\partial \overset{1}{A_i}}{\partial \mu} = - \sum_{j=1}^{L} \left[\overset{1}{A_i}, \overset{2}{\text{tr}} \left(\overset{2}{A_j} \sum_{\substack{A,B=0 \\ (A,B)\neq(0,0)}}^{K-1} \mathcal{Z}_{AB}(z_i - z_j) \overset{1}{\sigma}_{AB} \overset{2}{\sigma}^{AB} \right) \right]. \tag{4.6}$$

The tau-function of this system is defined as the generating function of the following Hamiltonians:

$$H_i = \frac{1}{4\pi i} \oint_{z_i} \text{tr}\, A^2(\gamma)\, d\gamma = \sum_{j=1, j\neq i}^{L} \sum_{\substack{A,B=0 \\ (A,B)\neq(0,0)}}^{K-1} A_j^{AB} A_{iAB} w_{AB}(z_i - z_j) \tag{4.7}$$

$$H_\mu = -\frac{1}{2\pi i} \oint_{a} \text{tr}\, A^2(\gamma)\, d\gamma = \frac{1}{2} \sum_{i,j=1}^{L} \sum_{\substack{A,B=0 \\ (A,B)\neq(0,0)}}^{K-1} A_j^{AB} A_{iAB} \mathcal{Z}_{AB}(z_i - z_j). \tag{4.8}$$

$$\frac{\partial \log \tau_{\text{Sch}}}{\partial z_i} = H_i \qquad \frac{\partial \log \tau_{\text{Sch}}}{\partial \mu} = H_\mu. \tag{4.9}$$

The following theorem shows how (analogously to the rational case [8]) solutions of the elliptic Schlesinger system (4.6) induce solutions of system (3.38) and (3.40).

Theorem 2. *Let $\mathcal{L}$ be a genus one covering of the λ-sphere with simple ramification points $P_1, \ldots, P_{2N}$, which have different λ-projections $\lambda_1, \ldots, \lambda_{2N}$. Consider a set of L points $\{Q_1, \ldots, Q_L\}$ on $\mathcal{L}$ such that their projections $\pi(Q_i)$ are independent of $\{\lambda_m\}$. Let ν be the Abel map (3.5) onto the fundamental domain of the covering, $\nu : \mathcal{L} \to T$. Consider the Schlesinger system (4.6) with $z_i = \nu(Q_i)$ and its solution $\{A_j(\{z_i\})\}_{j=1}^{L}$. Let $\Psi(\gamma, \{z_i\})$ be the corresponding solution of system (4.1). We can consider Ψ as a function on the covering $\mathcal{L}$ via the Abel map:*

$$\Psi(P) = \Psi(\nu(P), \{\nu(Q_i)\}). \tag{4.10}$$

Then

1. *the function $\Psi(P)$ satisfies the linear system (3.38) with J_m defined by*

$$\overset{1}{J_m} = -\alpha_m \sum_{j=1}^{L} \overset{2}{\text{tr}}(\overset{12}{r}(\gamma_m - z_j)\overset{2}{A_j}) \tag{4.11}$$

and, hence, J_m solve the system (3.40);

2. *the tau-function τ (3.41) of the system (3.40) is related to the tau-function τ_{Sch} (4.9) of the elliptic Schlesinger system according to*

$$\tau(\lambda_m) = \prod_{j=1}^{L} \left(\frac{\partial \nu}{\partial \lambda}(Q_j) \right)^{\text{tr}A_j^2/2} \tau_{\text{Sch}}\big(\{z_k\}|_{z_k=\nu(Q_k)}\big). \tag{4.12}$$

Remark. Formula (4.12) coincides with the one relating the tau-function of the rational system (2.11) and the tau-function of the Schlesinger system on the Riemann sphere, see [8].

Proof. Since the solution Ψ of (4.1) is defined on the space of branch coverings as in (4.10), we can differentiate it with respect to λ_m according to the chain rule:

$$\frac{d\Psi}{d\lambda_m} = \frac{\partial\Psi}{\partial\nu}\frac{\partial\nu}{\partial\lambda_m} + \sum_{j=1}^{L}\frac{\partial\Psi}{\partial z_j}\frac{\partial z_j}{\partial\lambda_m} + \frac{\partial\Psi}{\partial\mu}\frac{\partial\mu}{\partial\lambda_m}.$$

(Recall that μ is the b-period of the elliptic Riemann surface.) We differentiate $z_i = \nu(Q_i)$ according to the formula (3.11) for derivatives of ν and use also formulae (3.22), (4.1), (4.4), (4.5). Using the relation

$$w_{AB}(\gamma - z_j)(\rho(z_j - \gamma_m) - \rho(\gamma - \gamma_m)) + 2\pi i\mathcal{Z}_{AB}(\gamma - z_j) = -w_{AB}(\gamma_m - z_j)w_{AB}(\gamma - \gamma_m)$$

which can be proved by checking periodicity properties of both sides as $\gamma \to \gamma + 1$, $\gamma \to \gamma + \mu$ and behaviour at the pole $\gamma = \gamma_m$, we obtain:

$$\frac{d\overset{1}{\Psi}}{d\lambda_m} = -\alpha_m \overset{2}{\text{tr}}\left(\overset{12}{r}(\nu - \gamma_m)\sum_{j=1}^{L}\text{tr}(\overset{23}{r}(\gamma_m - z_j)\overset{3}{A}_j)\right)\overset{1}{\Psi}. \tag{4.13}$$

We single out the ν-dependent term and denote the rest by J_m:

$$\overset{2}{J}_m := -\alpha_m\sum_{j=1}^{L}\text{tr}(\overset{23}{r}(\gamma_m - z_j)\overset{3}{A}_j).$$

This leads to the system (3.38) and proves the first part of the theorem. For the second part, equality (4.12), we shall prove the following relation between the two tau-functions:

$$\frac{\partial\log\tau}{\partial\lambda_m} = \frac{\partial\log\tau_{\text{Sch}}}{\partial\lambda_m} + \sum_{j=1}^{L}\frac{\text{tr}\,A_j^2}{2}\frac{\partial\log\frac{\partial\nu}{\partial\lambda}}{\partial\lambda_m}\Bigg|_{\nu=z_j}. \tag{4.14}$$

This leads to (4.12) if one observes that

$$\frac{\partial\,\text{tr}\,A_j^2}{\partial\lambda_m} = 0$$

which follows from the Schlesinger system (4.6). To show (4.14) let us first note two auxiliary relations. The first one is

$$w_{AB}(z_i - z_j)(\rho(z_j - \gamma) - \rho(z_i - \gamma)) = w_{AB}(\gamma - z_j)w_{-A-B}(\gamma - z_i) - 2\pi i\mathcal{Z}_{AB}(z_i - z_j) \tag{4.15}$$

for any pair of non-equal indices i, j. This relation can be verified examining singularities and periodicity properties and then noting that at the point $\gamma = \frac{1}{2}(z_i + z_j)$ both sides are equal due to the equality

$$2w_{AB}(2\gamma)\rho(\gamma) = w_{AB}^2(\gamma) + 2\pi i\mathcal{Z}_{AB}(2\gamma)$$

which, in turn, can be proved by the same method. One can apply similar considerations to verify the second identity which we shall use

$$w_{AB}(\gamma)w_{-A-B}(\gamma) = 2\pi i\mathcal{Z}_{AB}(0) - \rho'(\gamma). \tag{4.16}$$

To show (4.14) we differentiate the tau-function of the elliptic Schlesinger system τ_{Sch} with respect to λ_m:

$$\frac{\partial\log\tau_{\text{Sch}}}{\partial\lambda_m} = \sum_{i=1}^{L}\frac{\partial\log\tau_{\text{Sch}}}{\partial z_i}\frac{\partial z_i}{\partial\lambda_m} + \frac{\partial\log\tau_{\text{Sch}}}{\partial\mu}\frac{\partial\mu}{\partial\lambda_m}.$$

Then we rewrite all the terms explicitly using (4.9), (4.7), (4.8), (3.11), (3.22) and simplify the obtained expression applying the auxiliary identity (4.15). Noting also that

$$\mathcal{Z}_{-A-B}(-\gamma) = \mathcal{Z}_{AB}(\gamma) \tag{4.17}$$

one arrives at the following expression:

$$\frac{\partial \log \tau_{\text{Sch}}}{\partial \lambda_m} = \alpha_m K \left(\sum_{\substack{i,j=1 \\ i<j}}^{L} \sum_{\substack{A,B=0 \\ (A,B)\neq(0,0)}}^{K-1} \epsilon^{AB} A_j^{AB} A_i^{-A-B} w_{AB}(\gamma_m - z_j) w_{-A-B}(\gamma_m - z_i) \right.$$

$$\left. + \pi i \sum_{i=1}^{L} \sum_{\substack{A,B=0 \\ (A,B)\neq(0,0)}}^{K-1} \epsilon^{AB} A_i^{AB} A_i^{-A-B} \mathcal{Z}_{AB}(0) \right). \tag{4.18}$$

The derivative in the second term of the right-hand side of (4.14) can be obtained using (3.11) as follows:

$$\frac{\partial \log \frac{\partial v}{\partial \lambda}}{\partial \lambda_m} = \frac{\partial}{\partial \lambda} \left(\frac{\partial v}{\partial \lambda_m} \right) \Big/ \frac{\partial v}{\partial \lambda} = \frac{\partial}{\partial v} \left(\frac{\partial v}{\partial \lambda_m} \right)$$

hence

$$\frac{\partial \log \frac{\partial v}{\partial \lambda}}{\partial \lambda_m} \bigg|_{v=z_j} = -\alpha_m \rho'(z_j - \gamma_m). \tag{4.19}$$

A certain simplification using the second auxiliary identity (4.16) leads to the following expression for the right-hand side of (4.14):

$$\frac{\partial \log \tau_{\text{Sch}}}{\partial \lambda_m} + \sum_{i=1}^{L} \frac{\text{tr } A_i^2}{2} \frac{\partial \log \frac{\partial v}{\partial \lambda}}{\partial \lambda_m} \bigg|_{v=z_i}$$

$$= \frac{\alpha_m}{2} \left(\sum_{i,j=1}^{L} \sum_{\substack{A,B=0 \\ (A,B)\neq(0,0)}}^{K-1} K \epsilon^{AB} A_j^{AB} A_i^{-A-B} w_{AB}(\gamma_m - z_j) w_{-A-B}(\gamma_m - z_i) \right) \tag{4.20}$$

which is nothing but $\text{tr}(J_m^2)/2\alpha_m$, where J_m are given by (4.11). Thus the right-hand side of (4.14) is equal to $(\log \tau)_{\lambda_m}$, where the tau-function τ is defined by (3.41). □

5. Trigonometric degeneration of the elliptic coverings and corresponding integrable systems

Here we describe the trigonometric version of system (3.40), obtained by a degeneration of the covering $\mathcal{L}$. Further, as an illustration, we consider the two-fold covering when all coefficients of the obtained system can be computed explicitly.

Set the matrix dimension K of the system to be 2. An elliptic N-fold covering has, according to the Riemann–Hurwitz formula, $2N$ branch points (recall that we assume them to be simple and distinct). If we let one branch cut degenerate (i.e. we let two ramification points connected by a branch cut tend to each other), the elliptic covering turns into a rational one with $2N - 2$ ramification points and a double point remaining from the degenerate branch cut.

Assume that the points P_{2N-1} and P_{2N} are connected by a branch cut $[P_{2N-1}, P_{2N}]$. Moreover, choose the basic a-cycle on $\mathcal{L}$ to surround this branch cut. Consider $\{\lambda_m\}_{m=1}^{2N-2}$ as independent variables and λ_{2N-1} and λ_{2N} as fixed parameters. Take the limit $\lambda_{2N-1}, \lambda_{2N} \to \lambda_Q$ with λ_Q independent of $\{\lambda_m\}_{m=1}^{2N-2}$. Then the branch cut $[P_{2N-1}, P_{2N}]$ degenerates and the elliptic curve $\mathcal{L}$ turns into the rational curve $\mathcal{L}_0$ with two marked points Q_1 and Q_2 (a double point) which lie on different sheets of $\mathcal{L}_0$ and have the same projection on the λ-plane:

$$\pi(Q_1) = \pi(Q_2) = \lambda_Q.$$

The basic a-cycle on $\mathcal{L}$ turns into a contour on $\mathcal{L}_0$ surrounding one of the points Q_1 or Q_2. Suppose that it surrounds Q_1 in the positive direction. Denote by $\zeta(P)$, $P \in \mathcal{L}_0$, the one-to-one map from the genus zero covering $\mathcal{L}_0$ with ramification points $P_1, \ldots, P_{2N-2}$ to the Riemann sphere; for simplicity we fix this map by the requirement $\zeta : \infty^{(0)} \to \infty$ such that in a neighbourhood of $\infty^{(0)}$

$$\zeta(\lambda) = \lambda + o(1). \tag{5.1}$$

Denote the images of points Q_1 and Q_2 on the Riemann sphere by κ_1 and κ_2, respectively:

$$\kappa_1 = \zeta(Q_1) \qquad \kappa_2 = \zeta(Q_2). \tag{5.2}$$

The holomorphic differential $\mathbf{v}(P)$ degenerates to the meromorphic on $\mathcal{L}_0$ differential $\mathbf{v}_0$ with the simple poles at Q_1 and Q_2 and residues $1/2\pi\mathrm{i}$ and $-1/2\pi\mathrm{i}$, respectively. This differential can be written in terms of the coordinate ζ as follows:

$$\mathbf{v}_0(\zeta) = \frac{1}{2\pi\mathrm{i}} \left(\frac{1}{\zeta - \kappa_1} - \frac{1}{\zeta - \kappa_2} \right) \mathrm{d}\zeta. \tag{5.3}$$

The b-period μ of the Riemann surface $\mathcal{L}$ in the limit $P_{2N-1} \to P_{2N}$ has the following behaviour:

$$\mu = \frac{1}{\pi\mathrm{i}} \log|\lambda_{2N-1} - \lambda_{2N}| + O(1) \tag{5.4}$$

i.e. $\mu \to +\mathrm{i}\infty$ in this limit, and the fundamental domain $T = \mathbb{C}/\{1, \mu\}$ of the covering $\mathcal{L}$ turns into a cylinder. The map ν (3.5) now maps the degenerate covering $\mathcal{L}_0$ onto the cylinder in the γ-sphere:

$$\nu(P) = \int_{\infty^{(0)}}^{P} \mathbf{v}_0 = \frac{1}{2\pi\mathrm{i}} \int_{\infty}^{\zeta} \left(\frac{1}{\zeta - \kappa_1} - \frac{1}{\zeta - \kappa_2} \right) \mathrm{d}\zeta = \frac{1}{2\pi\mathrm{i}} \log \frac{\zeta - \kappa_1}{\zeta - \kappa_2}. \tag{5.5}$$

From the definition (3.7) of the Jacobi theta-functions, we deduce the behaviour of the logarithmic derivative $\rho(\gamma)$ of $\theta_1 = \theta\left[\frac{1}{2}, \frac{1}{2}\right]$ as $\mu \to +\mathrm{i}\infty$:

$$\rho(\gamma) \to \pi \cot \pi\gamma \tag{5.6}$$

and therefore,

$$\rho'(\gamma) \to -\frac{\pi^2}{\sin^2 \pi\gamma}. \tag{5.7}$$

Similarly, the r-matrix becomes in this limit (for the matrix dimension $K = 2$):

$$\overset{12}{r}(\gamma) \to \overset{12}{r_0}(\gamma) = \frac{1}{2} \frac{\pi}{\sin \pi\gamma} \overset{1}{\sigma}_1 \overset{2}{\sigma}_1 + \frac{1}{2} \frac{\pi}{\sin \pi\gamma} \overset{1}{\sigma}_2 \overset{2}{\sigma}_2 + \frac{1}{2}\pi \cot \pi\gamma \overset{1}{\sigma}_3 \overset{2}{\sigma}_3 \tag{5.8}$$

where we use the Pauli basis $\{\sigma_i\}_{i=1}^{3}$ (3.48); r_0 is the so-called trigonometric r-matrix. Differential equations (3.23)–(3.26) for $\{\gamma_m\}_{m=1}^{2N-2}$ (images of non-degenerate ramification points $P_1, \ldots, P_{2N-2}$ under the map ν (5.5)) take the form (for $m \neq n$):

$$\frac{\partial \gamma_n}{\partial \lambda_m} = -\pi \alpha_m^0 [\cot \pi(\gamma_n - \gamma_m) + \cot \pi\gamma_m] \tag{5.9}$$

$$\frac{\partial \gamma_m}{\partial \lambda_m} = \pi \sum_{n=1,n\neq m}^{2N-2} \alpha_n^0 [\cot \pi (\gamma_m - \gamma_n) + \cot \pi \gamma_n]$$

$$\frac{\partial \alpha_n^0}{\partial \lambda_m} = 2\pi^2 \frac{\alpha_n^0 \alpha_m^0}{\sin^2 \pi (\gamma_n - \gamma_m)} \tag{5.10}$$

$$\frac{\partial \alpha_m^0}{\partial \lambda_m} = -2\pi^2 \sum_{n=1,n\neq m}^{2N-2} \frac{\alpha_n^0 \alpha_m^0}{\sin^2 \pi (\gamma_n - \gamma_m)}$$

where by α_m^0 we denoted the analogue of the coefficient α_m in the degenerate case:

$$\alpha_m^0 = \frac{1}{2} v_{0m}^2 = \frac{1}{2} \left[\left. \frac{\mathbf{v_0}(P)}{d\sqrt{\lambda - \lambda_m}} \right|_{P=P_m} \right]^2 \qquad m = 1, \ldots, 2N - 2. \tag{5.11}$$

Remark 1. Differential equations (5.9) can be obtained directly from the form (5.5) of the map ν using the fact that the map ζ satisfies equations (2.9) with $\beta = 1, c_n = 0$.

Remark 2. The system (5.9), (5.10) after a simple change of variables coincides with equations for characteristic speeds of the system of hydrodynamic type to which the Boyer–Finley equation (self-dual Einstein equation with one Killing vector) $U_{xy} = (e^U)_{tt}$ reduces [13].

The linear system for the matrix Ψ is written now via the trigonometric r-matrix r_0:

$$\frac{d\overset{1}{\Psi}(P)}{d\lambda_m} = \mathrm{tr}\,(\overset{2}{r}\,\overset{12}{0}\,(\nu(P) - \gamma_m)\,\overset{2}{J_m})\,\overset{1}{\Psi}(P) \tag{5.12}$$

$m = 1, \ldots, 2N - 2$. Then, the trigonometric version of system (3.40) for $J_m = J_m^1 \sigma_1 + J_m^2 \sigma_2 + J_m^3 \sigma_3$ (for notation see (3.49)) gives the compatibility condition of the above linear system:

$$\frac{\partial J_m^1}{\partial \lambda_n} = \frac{\alpha_n^0 \pi^2}{\sin^2 \pi (\gamma_m - \gamma_n)} J_m^1 + \frac{\alpha_m^0 \pi^2 \cos \pi (\gamma_m - \gamma_n)}{\sin^2 \pi (\gamma_m - \gamma_n)} J_n^1$$
$$+ \frac{2\pi \mathrm{i}}{\sin \pi (\gamma_m - \gamma_n)} \left(J_m^2 J_n^3 \cos \pi (\gamma_m - \gamma_n) - J_m^3 J_n^2 \right)$$

$$\frac{\partial J_m^2}{\partial \lambda_n} = \frac{\alpha_n^0 \pi^2}{\sin^2 \pi (\gamma_m - \gamma_n)} J_m^2 + \frac{\alpha_m^0 \pi^2 \cos \pi (\gamma_m - \gamma_n)}{\sin^2 \pi (\gamma_m - \gamma_n)} J_n^2 \tag{5.13}$$
$$+ \frac{2\pi \mathrm{i}}{\sin \pi (\gamma_m - \gamma_n)} \left(J_m^3 J_n^1 - J_m^1 J_n^3 \cos \pi (\gamma_m - \gamma_n) \right)$$

$$\frac{\partial J_m^3}{\partial \lambda_n} = \frac{\alpha_n^0 \pi^2}{\sin^2 \pi (\gamma_m - \gamma_n)} J_m^3 + \frac{\alpha_m^0 \pi^2}{\sin^2 \pi (\gamma_m - \gamma_n)} J_n^3 + \frac{2\pi \mathrm{i}}{\sin \pi (\gamma_m - \gamma_n)} \left(J_m^1 J_n^2 - J_m^2 J_n^1 \right)$$

the indices m and n are different and range in the set $\{1, \ldots, 2N - 2\}$.

All the involved coefficients can be explicitly computed if we start with the two-fold elliptic covering. After the degeneration we get a rational covering $\mathcal{L}_0$ with two ramification points P_1 and P_2 (with the λ-projections λ_1 and λ_2) and the marked points with the projection λ_Q independent of λ_1 and λ_2. The one-to-one map ζ from this covering to the Riemann sphere which satisfies condition (5.1) has the following form:

$$\zeta(P) = \frac{1}{2}\left(\lambda + \frac{\lambda_1 + \lambda_2}{2} + \sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)}\right) \tag{5.14}$$

where $\lambda = \pi(P)$, the projection of the point P on the base of the covering. Knowing the expression for the map ζ allows us to find the images γ_1, γ_2 of the non-degenerate ramification points P_1, P_2 under the map $\nu(P)$ since it can be explicitly integrated (see(5.5)):

$$\gamma_m = \frac{1}{2\pi i} \log \frac{\zeta_m - \kappa_1}{\zeta_m - \kappa_2} \tag{5.15}$$

where $\zeta_m = \zeta(P_m)$, $m = 1, 2$ are the images on the Riemann sphere of ramification points. One can find them from the form (5.14) of the map ζ:

$$\zeta_1 = \zeta(\lambda_1) = \frac{3\lambda_1 + \lambda_2}{4} \qquad \zeta_2 = \zeta(\lambda_2) = \frac{\lambda_1 + 3\lambda_2}{4}.$$

For the ζ-images $\kappa_{1,2}$ of points Q_1 and Q_2 (5.2) we have

$$\kappa_{1,2} = \frac{1}{2}\left(\lambda_Q + \frac{\lambda_1 + \lambda_2}{2} \pm \sqrt{(\lambda_Q - \lambda_1)(\lambda_Q - \lambda_2)}\right).$$

Now one can easily see from (5.15) that

$$e^{2\pi i\gamma_1} = \frac{\sqrt{\lambda_1 - \lambda_Q} - \sqrt{\lambda_2 - \lambda_Q}}{\sqrt{\lambda_1 - \lambda_Q} + \sqrt{\lambda_2 - \lambda_Q}} = -e^{2\pi i\gamma_2} \tag{5.16}$$

and, therefore, $\gamma_1 - \gamma_2 = \pm 1/2$. The same conclusion can be drawn if one observes that $\gamma_1 - \gamma_2$ is equal to one half of the integral over the a-period of the differential $\mathbf{v}$ (see definition (3.5) of the map ν). The sign of the difference $\gamma_1 - \gamma_2$ is determined by the choice of direction of the a-cycle.

It remains to calculate one more ingredient of system (3.40) for J_m, namely the coefficients $\alpha_{1,2}^0$ (5.11). Denoting by $v_0(x)$ a locally defined function such that $\mathbf{v_0} = v_0(x)\, dx$ (x being a local parameter on the covering), from the relation

$$v_0(x)\, dx = \frac{1}{2\pi i}\left(\frac{1}{\zeta - \kappa_1} - \frac{1}{\zeta - \kappa_2}\right) d\zeta$$

we deduce that

$$\frac{d\zeta}{dx}(\lambda_m) = 2\pi i v_{0m} \frac{(\zeta_m - \kappa_1)(\zeta_m - \kappa_2)}{\kappa_1 - \kappa_2} \qquad m = 1, 2.$$

From the explicit form (5.14) of the map $\zeta(P)$ one can compute the coefficients $(d\zeta/dx)(\lambda_m)$ of expansion of $\zeta(P)$ in neighbourhoods of ramification points P_1, P_2. Then we obtain the expressions for $\alpha_m^0 = \frac{1}{2}v_{0m}^2$ ($m = 1, 2$):

$$\alpha_1^0 = -\frac{1}{2\pi^2}\frac{\lambda_2 - \lambda_Q}{\lambda_1 - \lambda_Q}\frac{1}{\lambda_1 - \lambda_2} \qquad \alpha_2^0 = -\frac{1}{2\pi^2}\frac{\lambda_1 - \lambda_Q}{\lambda_2 - \lambda_Q}\frac{1}{\lambda_2 - \lambda_1}.$$

In the limit $\lambda_Q \to \infty$, summarizing all the above calculations, we get from (5.13) the system of equations for $J_1 = J_1^1\sigma_1 + J_1^2\sigma_2 + J_1^3\sigma_3$:

$$\frac{\partial J_1^1}{\partial \lambda_2} = \frac{1}{2}\frac{1}{\lambda_1 - \lambda_2}J_1^1 - 2\pi i J_1^3 J_2^2$$

$$\frac{\partial J_1^2}{\partial \lambda_2} = \frac{1}{2}\frac{1}{\lambda_1 - \lambda_2}J_1^2 + 2\pi i J_1^3 J_2^1$$

$$\frac{\partial J_1^3}{\partial \lambda_2} = \frac{1}{2}\frac{1}{\lambda_1 - \lambda_2}(J_1^3 - J_2^3) + 2\pi i(J_1^1 J_2^2 - J_1^2 J_2^1)$$

and the similar system for J_2:

$$\frac{\partial J_2^1}{\partial \lambda_1} = \frac{1}{2} \frac{1}{\lambda_2 - \lambda_1} J_2^1 + 2\pi i J_2^3 J_1^2$$

$$\frac{\partial J_2^2}{\partial \lambda_1} = \frac{1}{2} \frac{1}{\lambda_2 - \lambda_1} J_2^2 - 2\pi i J_2^3 J_1^1$$

$$\frac{\partial J_2^3}{\partial \lambda_1} = \frac{1}{2} \frac{1}{\lambda_1 - \lambda_2} \left(J_1^3 - J_2^3 \right) + 2\pi i \left(J_1^1 J_2^2 - J_1^2 J_2^1 \right).$$

Acknowledgment

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